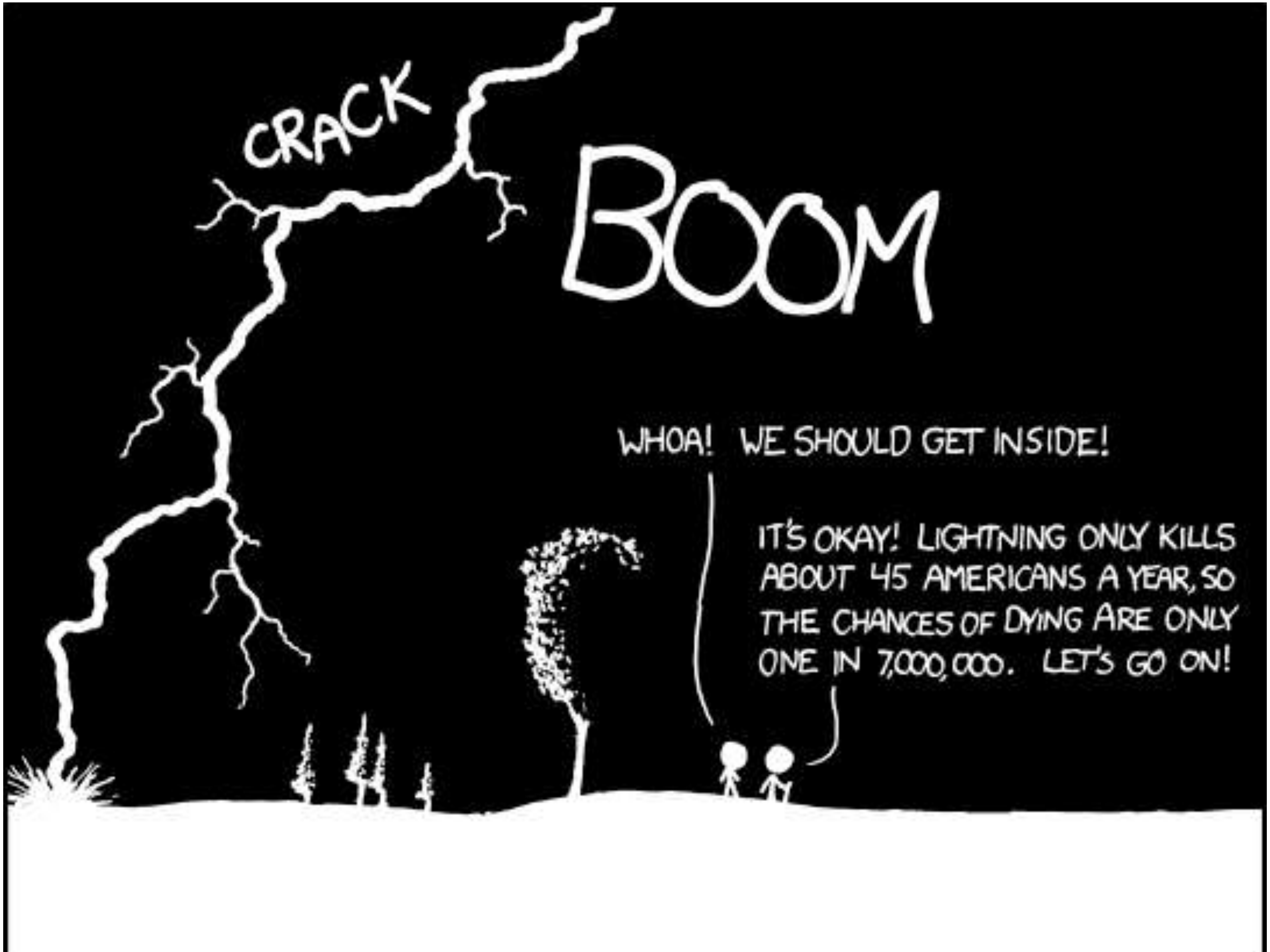
Stat 515: Introduction to Statistics

Chapter 3



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT *STATISTIC* IS ONE IN SIX.

Xkcd.com

Topics Up to This Point

- Chapter One
  - Samples vs Populations
  - Data types
  - Sampling Methods and bias
- Chapter Two
  - Representing data – graphical and numerical
  - Measures of Central Tendency and Dispersion
  - Five number summary & Box plots
  - Outliers

Now… Probability

- This is very different but not unrelated
- This will help segue us into inferential statistics by allowing us to talk about the probability that an event will or will not happen and, more importantly, how likely it is to see a particular event given certain information

Introduction

- A lot of information really fast! Really, it's almost the whole chapter in 8 minutes!*
  - https://www.youtube.com/watch?v=YpvE0Co66nU
- Spinner probabilities
  - https://www.youtube.com/watch?v=QpfMwA0z_1Y
- No Punting
  - https://www.youtube.com/watch?v=AGDaOJAYHfo

Probability - Definitions

- **Random Experiment –** Any action or observable experiment where the singular outcome is random
  - Possible outcomes are known but it is uncertain which will occur for any given observation.
  - **Examples:** Flipping a coin, rolling a die, etc

Probability - Definitions

- **Sample space –** the set of all possible outcomes in an experiment
  - **Examples**
    - Rolling a die: S={1,2,3,4,5,6}
    - Flipping a coin: S={Heads, Tails}

Probability - Definitions

- **Sample Point** – We refer to a particular outcome occurring as an sample point or event
  - Examples: **A**=we rolled a 4, **A**= flipped head

- **An Event** – a collection of sample points (note this could be a collection of one)

Probability - Definitions

- **Example: Rolling a die**
  - **Sample Space =** S = {1,2,3,4,5,6}
  - Let **event A =** roll a multiple of three = {3,6}
    - 3 and 6 are **sample points**

Introduction to Probability

- **Probability –** The way we quantify uncertainty in random experiments

- **Randomness –** Possible outcomes are known but it is uncertain which will occur for any given observation.
  - **Note:** Random phenomena is highly variable so we look at patterns in the long run

Law of Large Numbers

- (**LLN 1**) – As the sample size increases the sample estimates ($\bar{x}$ or $\hat{p}$) approach the population values (μ or σ)

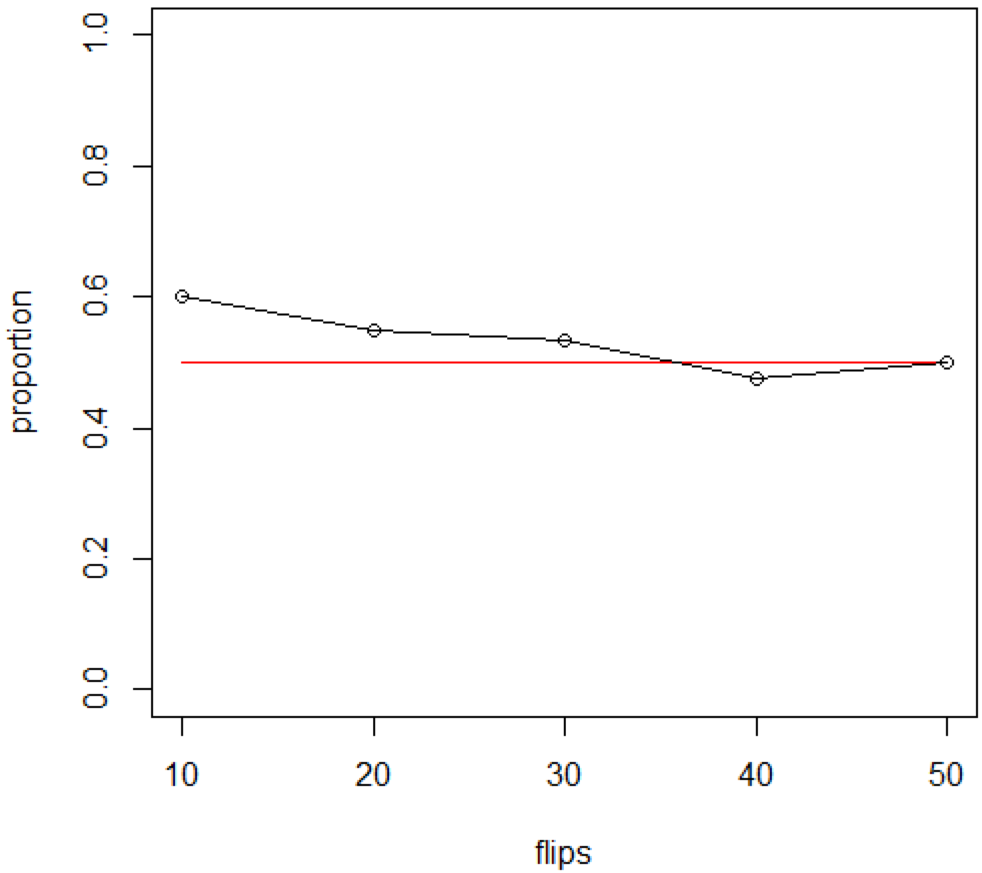
- (**LLN 2**) – As the number of trials increase the **proportion** of occurrences of any given outcome approaches the probability in the long run.

Simulation of Coin Flips

- 10 flips: 6 heads were flipped
  - $Total\ proportion\ = \frac{x}{n} = \frac{6}{10} = .60 = 60\%\ heads$
- 10 more flips: 5 heads were flipped
  - $Total\ proportion\ = \frac{x}{n} = \frac{5+6}{10+10} = \frac{11}{20} = .55 = 55\%\ heads$
- 10 more flips: 5 heads were flipped
  - $Total\ proportion\ = \frac{x}{n} = \frac{11+5}{20+10} = \frac{16}{30} = .5333 = 53.33\%\ heads$
- 10 more flips: 3 heads were flipped
  - $Total\ proportion\ = \frac{x}{n} = \frac{16+3}{30+10} = \frac{19}{40} = .475 = 47.5\%\ heads$
- 10 more flips: 6 heads were flipped
  - $Total\ proportion\ = \frac{x}{n} = \frac{19+6}{40+10} = \frac{25}{50} = .5 = 50\%\ heads$

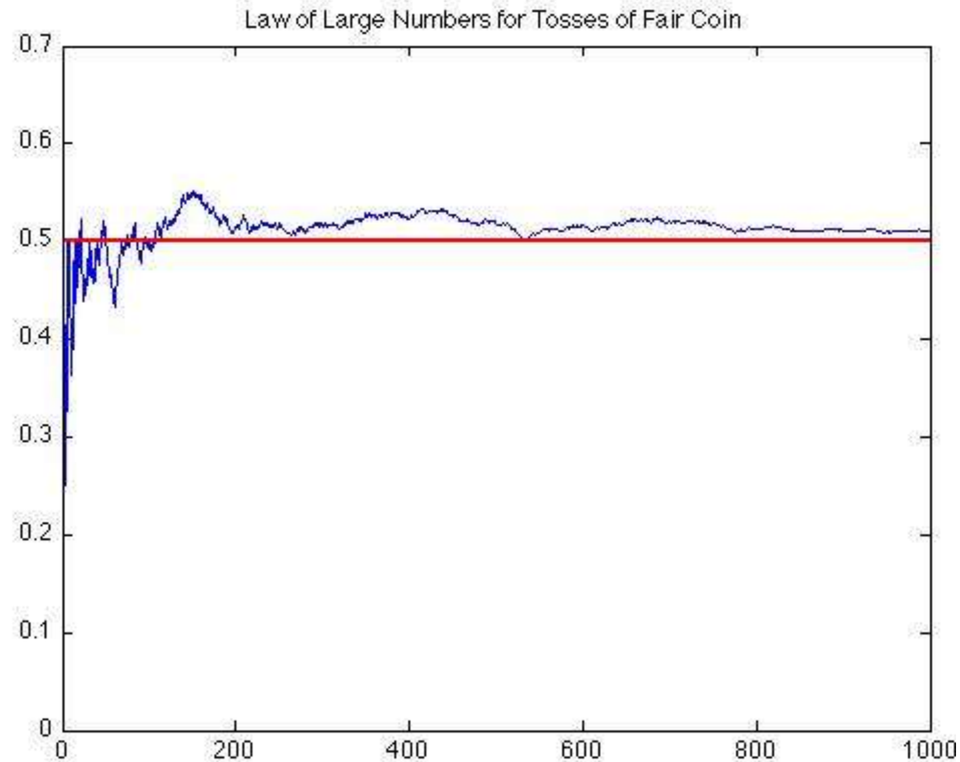
Simulation of Coin Flips

- (**LLN**) – As the number of flips increase the **proportion** of heads approaches the **probability** of seeing a heads, P(heads)=.5, which is the red line.



A Bigger Example of the Law of Large Numbers

- Continuing the experiment to 1000 flips:
- At first the proportion is all over the place – you can see the large spikes in the graph
- Importantly, we see that the proportion of coins that landed on heads levels off and gets closer and closer to 50%, the probability, which is where we expect it to go 'in the long run!'



Long-run Probability

- The probability of a particular outcome is the proportion of times that the outcome would occur in the long-run, as our sample size grows unbounded.
  - Note: the probability is the way you already think about it – you just never knew you were doing ‘long run’ probability!

Probability – How to Calculate

- **Empirical Probability of A**

$$\widehat{P(A)} = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ Event\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$$

$= \sum P(a_i)$ where a_i are the sample points of A

- **Probability Properties:**
  - Probability cannot be negative
  - $0 \leq P(A) \leq 1$
  - The sum of the probability of all possible outcomes of a random experiment must equal one

Probability – How to Calculate 1

- **Probability** we roll a multiple of three:
- **Sample Space =** S = {1,2,3,4,5,6}

$$P(\widehat{A = \{3,}6\}) = \frac{N(A)}{N(S)}$$

$$= \frac{(Number\ of\ ways\ we\ can\ roll\ a\ 3\ or\ 6)}{(Total\ number\ of\ outcomes)} = \frac{2}{6}$$

$$= \frac{1}{3}$$

Probability – How to Calculate 2

- **Probability** we roll a multiple of three:
- **Sample Space =** S = {1,2,3,4,5,6}

$$P(A = \{3,6\}) = \sum P(a_i) \text{ where } a_i \text{ are sample points in A}$$
$$= P(A = \{3\}) + P(A = \{6\})$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Probability – How to Calculate a List

1. **Find the Sample Space S**
2. **List the sample points and their probability**
3. **Decide which sample points belong in the event of interest**
4. **Sum the probabilities of the sample points chosen in part three**

Ways to Count Complicated Events

Multiplication Rule of Counting

- If a task consists of a sequence of choices in which there are **p selections** for the first choice, **q selections** for the second choice, **r selections** for the third choice and so forth
- Then, together, the task can be done in (p*q*r*…) different ways

Multiplication Rule of Counting w/ Replacement

- A South Carolina license plate is three letters followed by three numbers. How many unique plates can they make in this format?
- 26 letters in the alphabet
- 10 single digit numbers
- 26*26*26*10*10*10 = 17,576,000 plates

Multiplication Rule of Counting w/o Replacement

- Let’s say to ease reading of plates South Carolina changes its license plate to be three letters followed by three numbers, but there can’t be any repeats of letters or digits. How many unique plates can they make in this format?
- 26 letters in the alphabet
- 10 single digit numbers
- 26*25*24*10*9*8 = 11,232,000 plates. Notice this is a lot less than before!

Multiplication Rule of Counting w/o Replacement

- 26*25*24*10*9*8 = 11,232,000 plates. Notice this is a lot less than before!
- Here, we see two instances where we multiply digits decreasing by one, in order. It isn't bad here because there are only three choices in each instance but what if license plates had more letters or digits?
  - To solve this we introduce **factorials**

Multiplication Rule of Counting w/o Replacement

- A **factorial** of a number $n \geq 0$ is defined as n! = n*(n-1)*(n-2)*…*3*2*1

- n! = n* (n-1)!

- 0! = 1
  - This is the trap door for the previous rule

Permutations

- A **permutation** is an ordered arrangement in which r objects are chosen from n distinct objects
  - $r \leq n$
  - No repetition

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutations

- For example, suppose we have a set of three letters: A, B, and C. We might ask how many ways we can arrange 2 letters from that set. Each possible arrangement would be an example of a permutation.
- The complete list of possible permutations would be: AB, AC, BA, BC, CA, and CB.
  - Here AB and BA are distinctly different!

Combinations

- A **combination** is collection, without regard to order, in which r objects are chosen from n distinct objects
  - $r \leq n$
  - No repetition

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!} $$

Combinations

- For example, suppose we have a set of three letters: A, B, and C. We might ask how many ways we can select 2 letters from that set. Each possible selection would be an example of a combination.
- The complete list of possible selections would be: AB, AC, and BC.
  - Here AB and BA are treated the same and just represented by AB

Permutations vs Combinations

- The distinction between a **combination** and a **permutation** has to do with the sequence or order in which objects appear.
- A **combination** focuses on the selection of objects without regard to the order in which they are selected.
- A **permutation**, in contrast, focuses on the arrangement of objects with regard to the order in which they are arranged.

Combinations Example

- At Wendy’s you can order a hamburger with the following toppings: Cheese, bacon, mayo, ketchup, mustard, pickles, onion, lettuce, tomato.
- **Q:** How many different burgers can you order at Wendy’s?
- **A:** To answer this question we will use combinations because the order the toppings go on doesn‘t matter

Combinations Example

- For any hamburger we have nine possible toppings – here are the easy parts
- **Q:** How many hamburgers can you make with zero toppings?
- **A:** Just one – the burger itself
- **Q:** How many hamburgers can you make with nine toppings?
- **A:** Just one – the burger with all nine toppings

Combinations Example

- So far we’ve found two possible hamburgers – all toppings and no toppings.
- **Q:** How many hamburgers can you make with one topping?
- **A:** Here we can find nine different hamburgers – each one with one of the nine toppings

Combinations Example

- So far we've found 11 possible hamburgers – all toppings, no toppings and 9 one-topping-each
- **Q:** How many other hamburgers can you make?
- **A:** Here is were it gets a little complicated - the left over combinations aren't as easy to think of.

Combinations Example

Number of Toppings	Number of Possible Burgers
0	1
1	9
2	nrow(combinations(9,2)) = 36
3	nrow(combinations(9,3)) = 84
4	nrow(combinations(9,4)) = 126
5	nrow(combinations(9,5)) = 126
6	nrow(combinations(9,6)) = 84
7	nrow(combinations(9,7)) = 36
8	nrow(combinations(9,8)) = 9
9	1
TOTAL	512

Combinations Example

- In each row of the previous table we found the number of possible hamburgers for that number of toppings
  - There are 84 different hamburgers with 2 toppings
- **In total we have 512 possible hamburgers**
  - **Note: we could have calculated this by considering each topping as a yes or a no (2 possibilities)**

$$2^{\#\, toppings} = 2^9 = 512$$

Permutations Example

Sheldon: Well, this sandwich is an unmitigated disaster. I asked for turkey and roast beef with lettuce and swiss on whole wheat.
Raj: What did they give you?
Sheldon: Turkey and roast beef with swiss and lettuce on whole wheat. It's the right ingredients, but in the wrong order. In a proper sandwich, the cheese is adjacent to the bread to create a moisture barrier against the lettuce. They might as well have dragged this thing through a car wash.

Permutations Example

- At Wendy's you can order a hamburger with the following toppings: Cheese, bacon, mayo, ketchup, mustard, pickles, onion, lettuce, tomato.
- **Q:** If my friend is insane and order matters, how many possible hamburgers are there?
- **A:** To answer this question we will use permutations because the order the toppings go on does matter

Permutations Example

- For any hamburger we have nine possible toppings – here are the easy parts
- **Q:** How many hamburgers can you make with zero toppings?
- **A:** Just one – the burger itself
- **Q:** How many hamburgers can you make with one topping?
- **A:** Nine – the burger with one topping has no ordering

Combinations Example

- So far we’ve found 10 possible hamburgers – all toppings and 9 one-topping-each
- **Q:** How many other hamburgers can you make?
- **A:** Here is were it gets a little complicated - the left over permutations aren’t as easy to think of.

Permutations Example

Number of Toppings	Number of Possible Burgers
0	1
1	9
2	nrow(permutations(9,2)) = 72
3	nrow(permutations(9,3)) = 504
4	nrow(permutations(9,4)) = 3,024
5	nrow(permutations(9,5)) = 15,120
6	nrow(permutations(9,6)) = 60,480
7	nrow(permutations(9,7)) = 181,440
8	nrow(permutations(9,8)) = 362,880
9	nrow(permutations(9,9)) = 362,880
TOTAL	986,410

Permutations Example

- In each row of the previous table we found the number of possible hamburgers for that number of toppings when order matters
  - There are 72 different hamburgers with 2 toppings
- **In total we have 986,410 possible hamburgers**
  - **Note: this is a lot more than the 512 combinations!**

Partitions Rule

- Suppose we want to partition a single set of N elements into k sets with n_1 items in set one, n_2 items in set two, …, n_k items in set k, such that $n_1 + n_2 + \cdots + n_k = N$
- The number of possible partitions is:

$$\frac{N!}{n_1!\, n_2! \ \ldots n_k!}$$

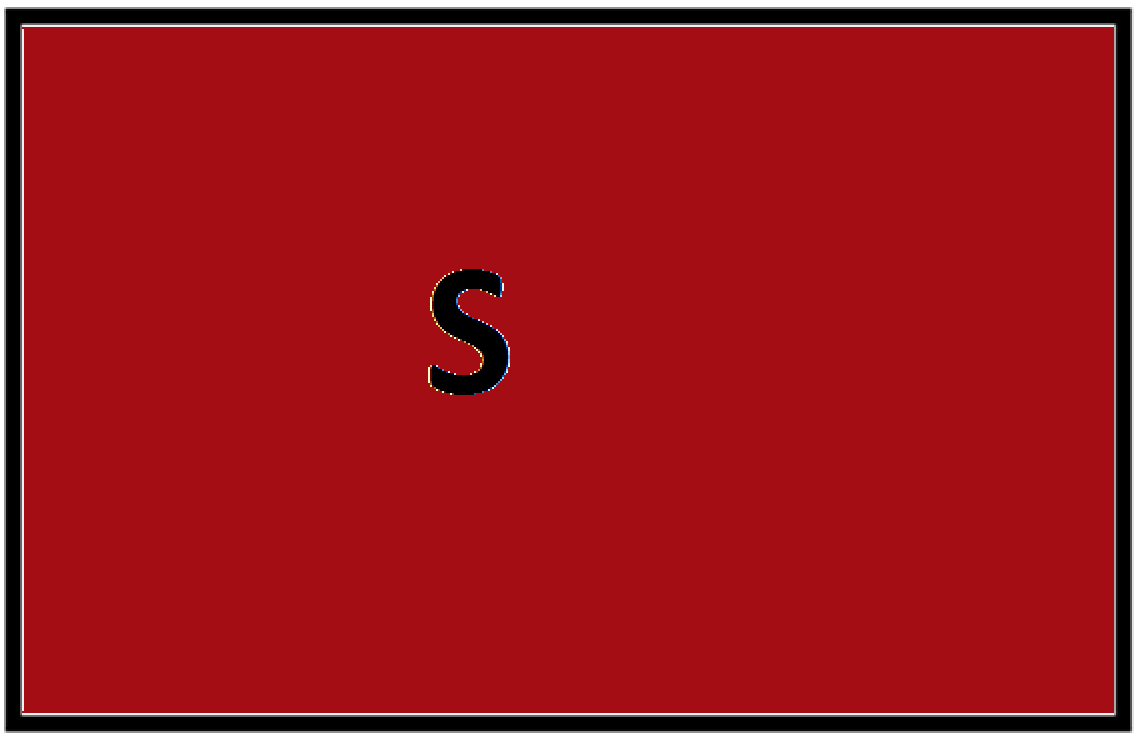
Partitions Rule : Example

- Suppose we have 20 students and we want to break them up into groups to solve four problems. The group sizes should be {2, 5, 5, 8} assigning less to the easier problems and more to the harder problems.
- The number of possible partitions is:

$$\frac{N!}{n_1!\,n_2!\ \ldots n_k!} = \frac{20!}{2!\,5!\,5!\,8!} = 2{,}095{,}133{,}040$$

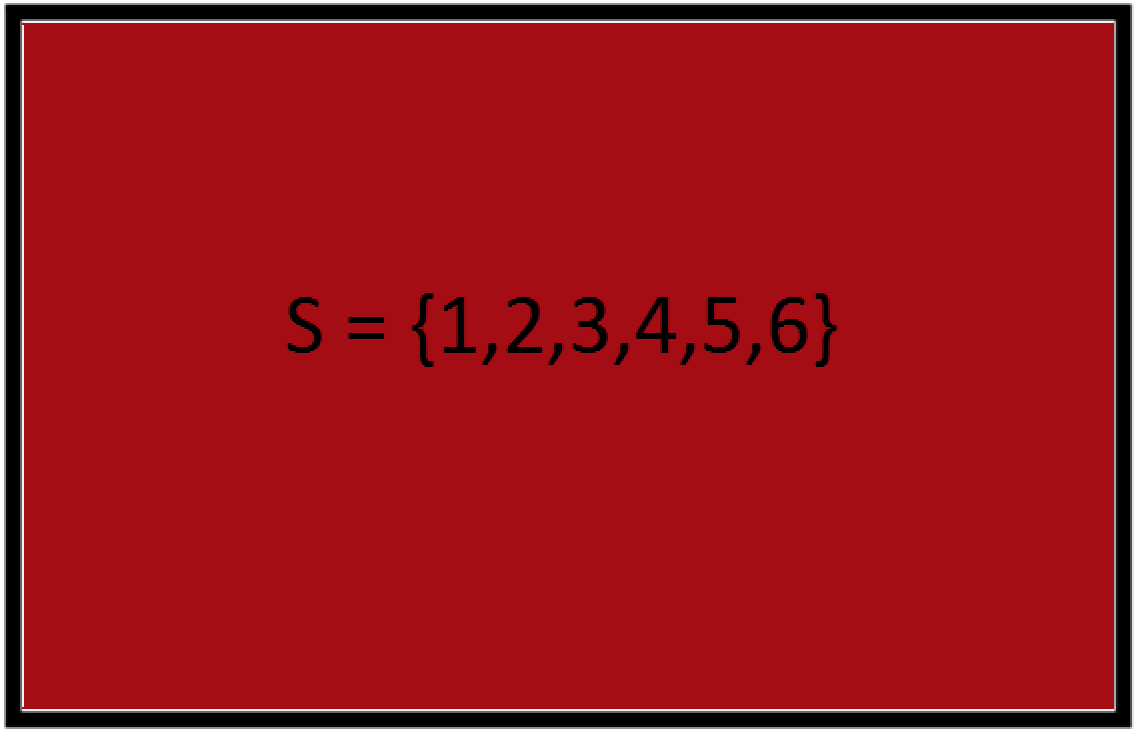
Adjectives for Experiments: Sample Space

- **The Sample Space** is the collection of all outcomes of a random experiment



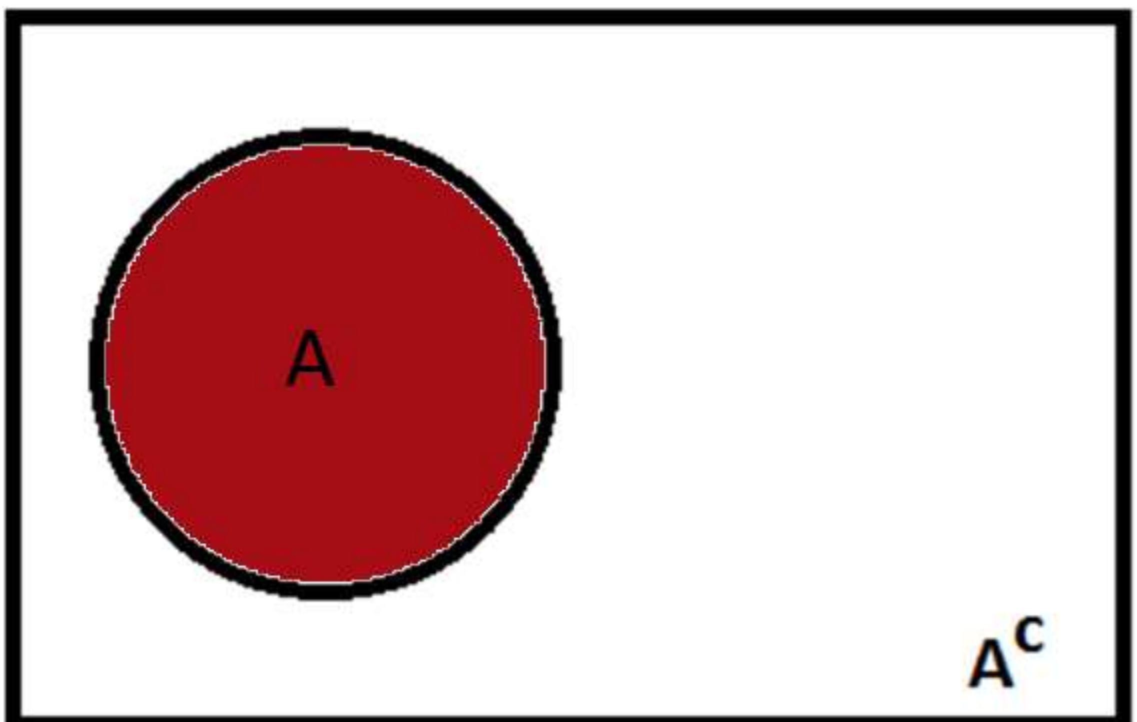
Adjectives for Experiments: Sample Space

- **The Sample Space** = $S = \{1,2,3,4,5,6\}$



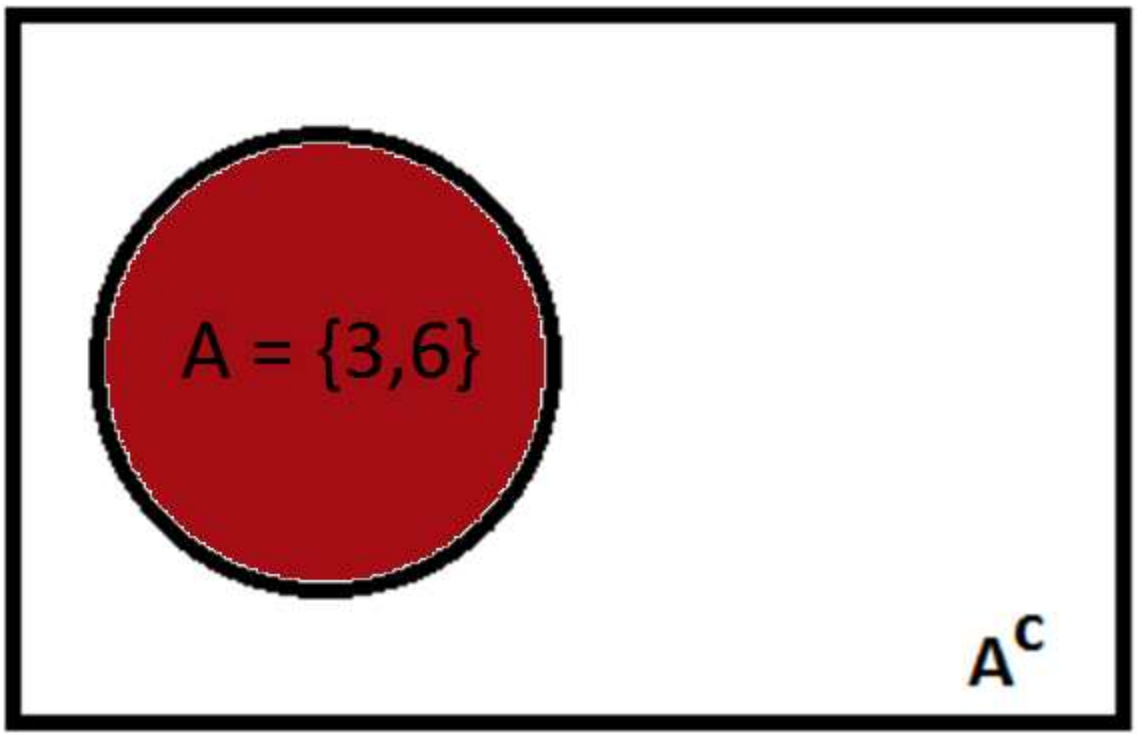
Adjectives for Experiments: Events

- **Event A** is the collection of all outcomes that belong to the event A



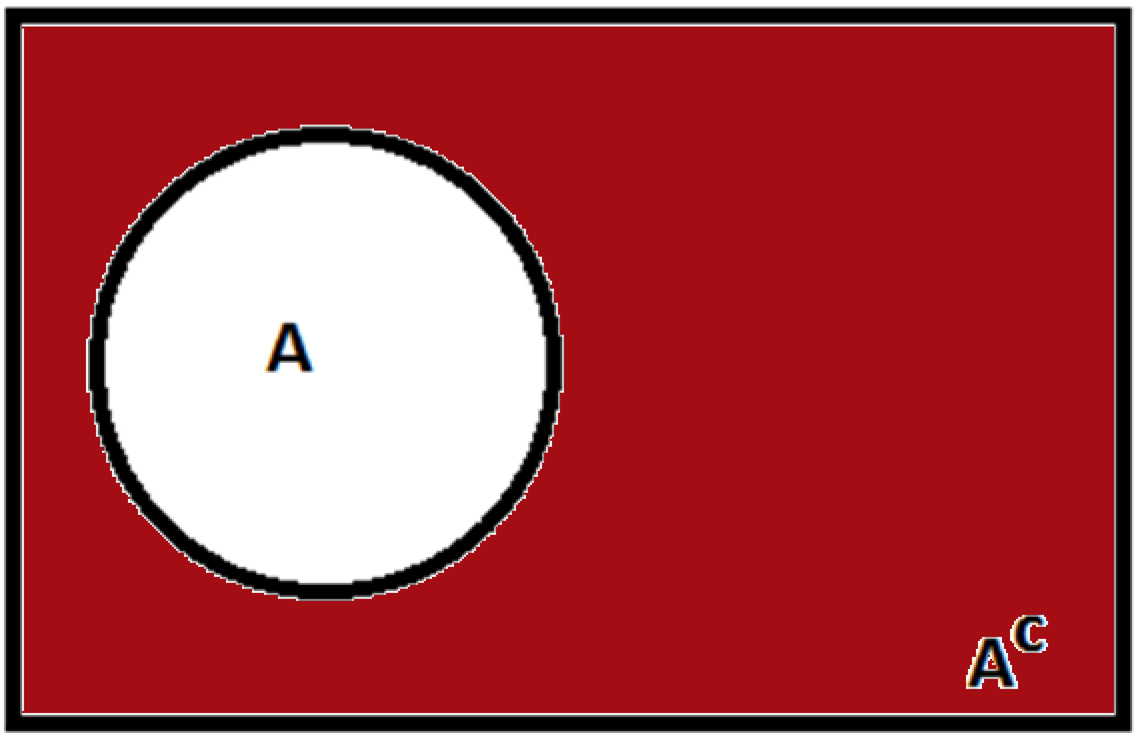
Adjectives for Experiments: Events

- **Event A** = {3,6}



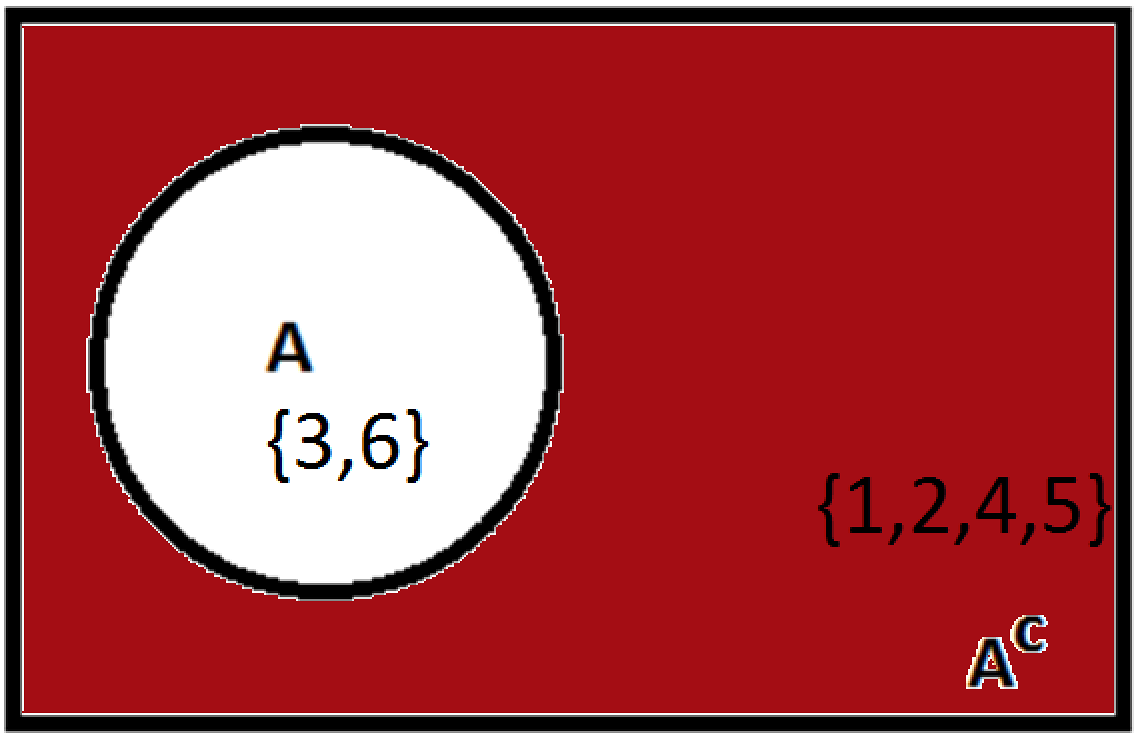
Adjectives for Experiments: Complement

- **The complement** of an event A is the set of all outcomes not in A



Adjectives for Experiments: Complement

- **The complement** of $\mathrm{A} = \mathrm{A}^c = \{1,2,4,5\}$



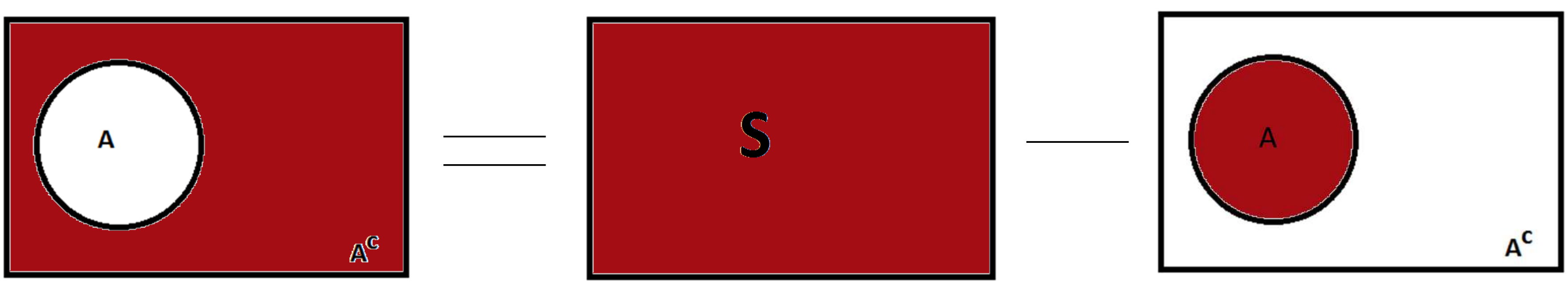
Probability: Complement Rule

- **Complement Rule:**
  - The probability of something not happening is 1 minus the probability of it happening

$$P(A^c) = P(S) - P(A)$$
$$P(A^c) = 1 - P(A)$$

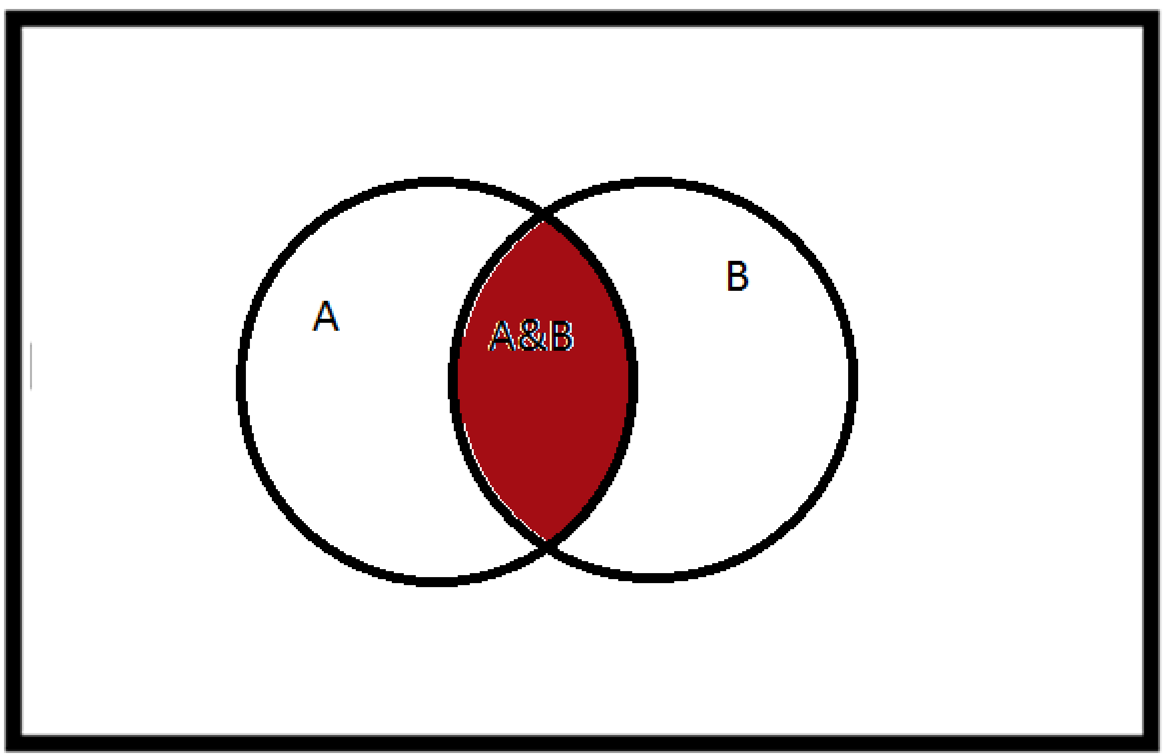
OR

$$P(A^c) + P(A) = 1$$



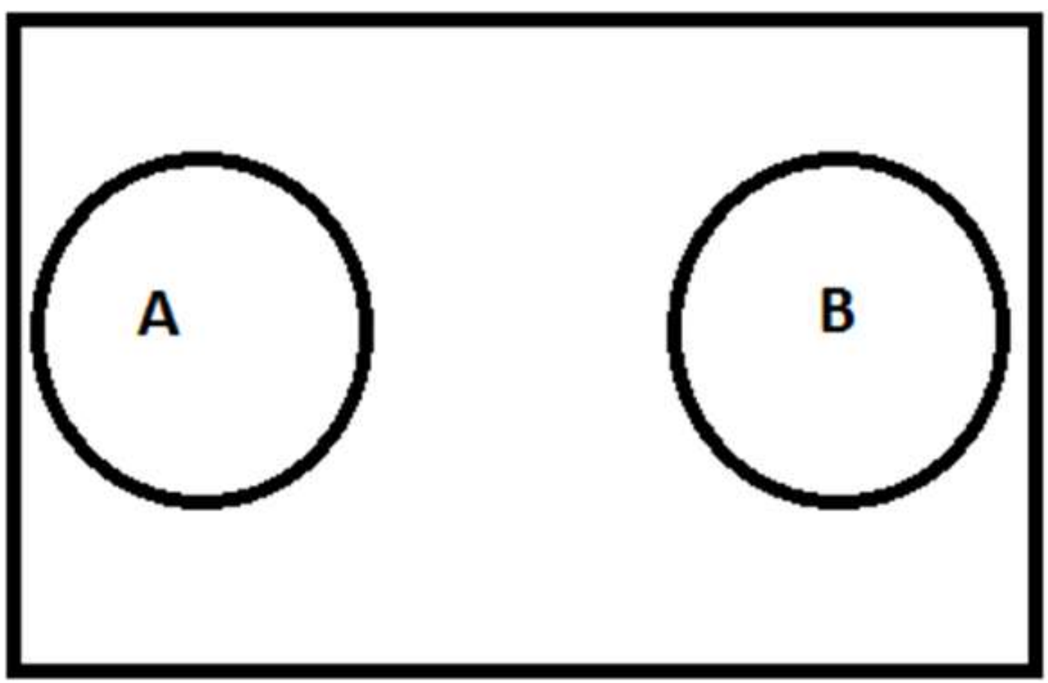
Adjectives for Experiments: Intersection

- The **intersection** of A and B consists of outcomes that are both in A and B



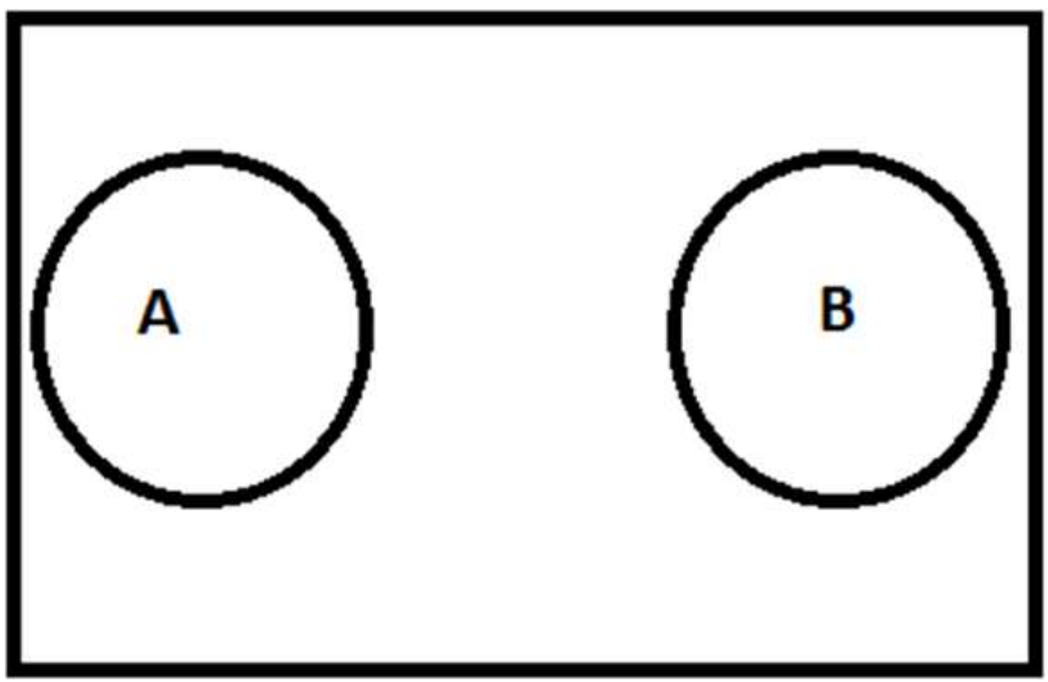
Adjectives for Experiments: Mutually Exclusive (Disjoint)

- Two events are **mutually exclusive (disjoint)** if they do not have any common outcomes
  - When two events have no **intersection**



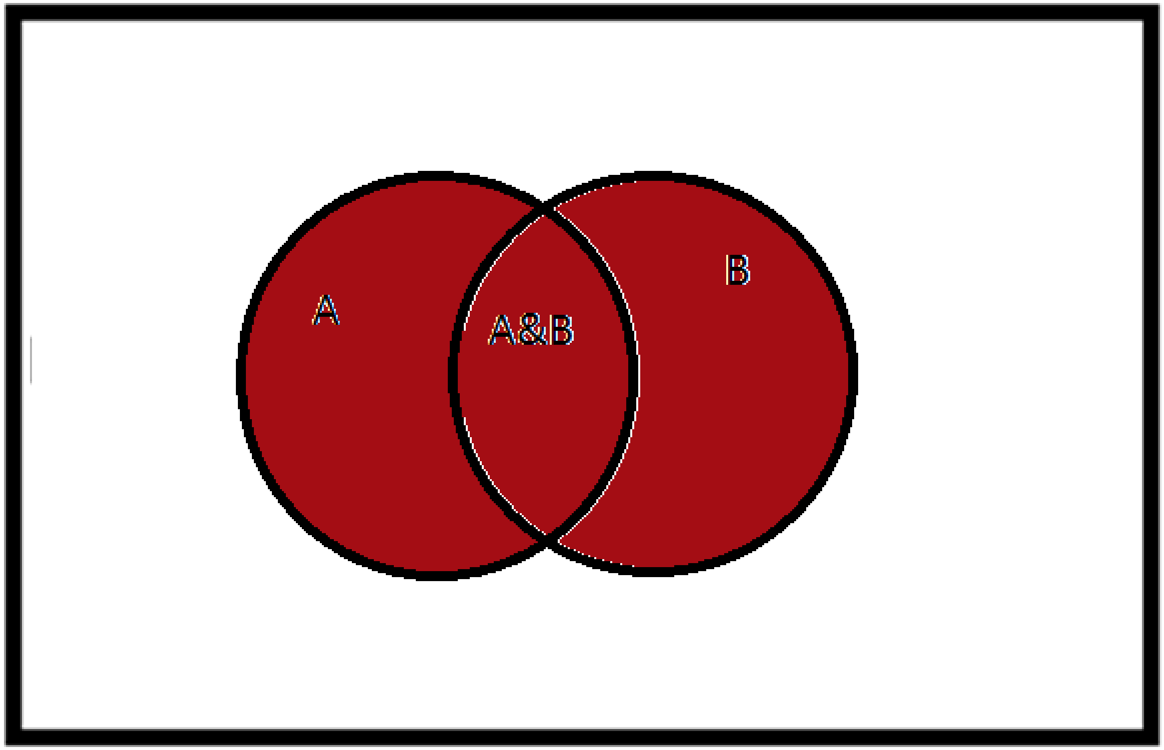
Probability: Mutually Exclusive (Disjoint)

- Two events are **mutually exclusive (disjoint)** if $\mathrm{P}(\mathrm{A\ and\ B}) = P(A \cap B) = 0$



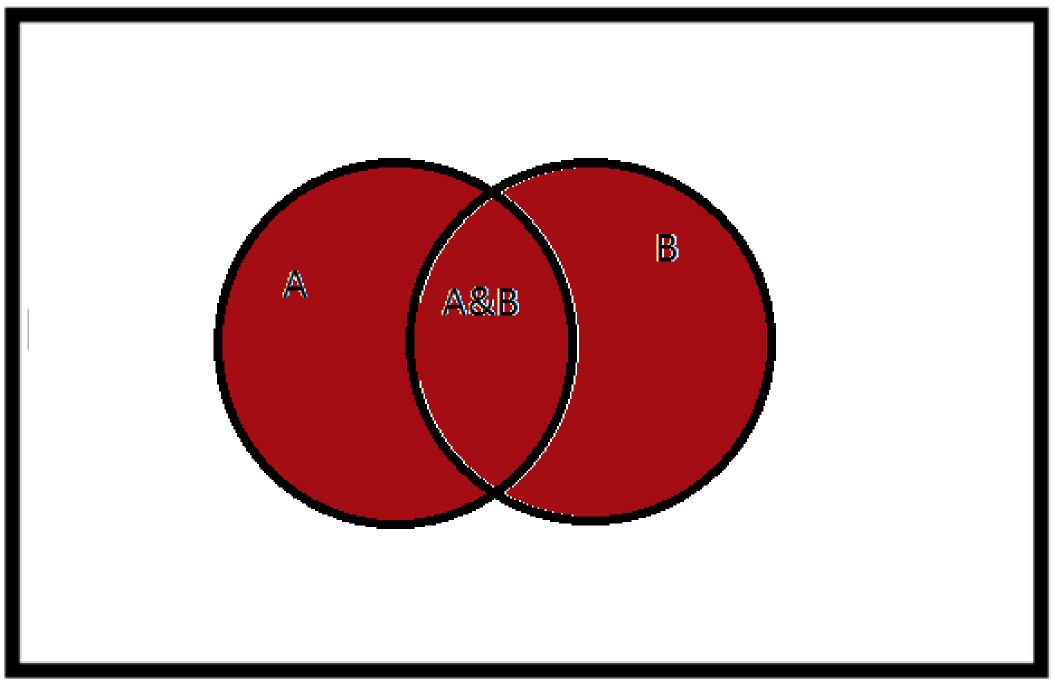
Adjectives for Experiments: Union

- The **union** of A and B consists of outcomes that are A or B or both



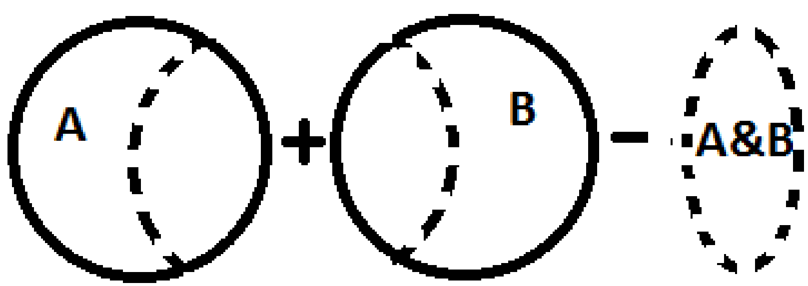
Probability: Additive Rule of Probability

- The **union** of A and B consists of outcomes that are A or B or both
- $P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Probability: Additive Rule of Probability

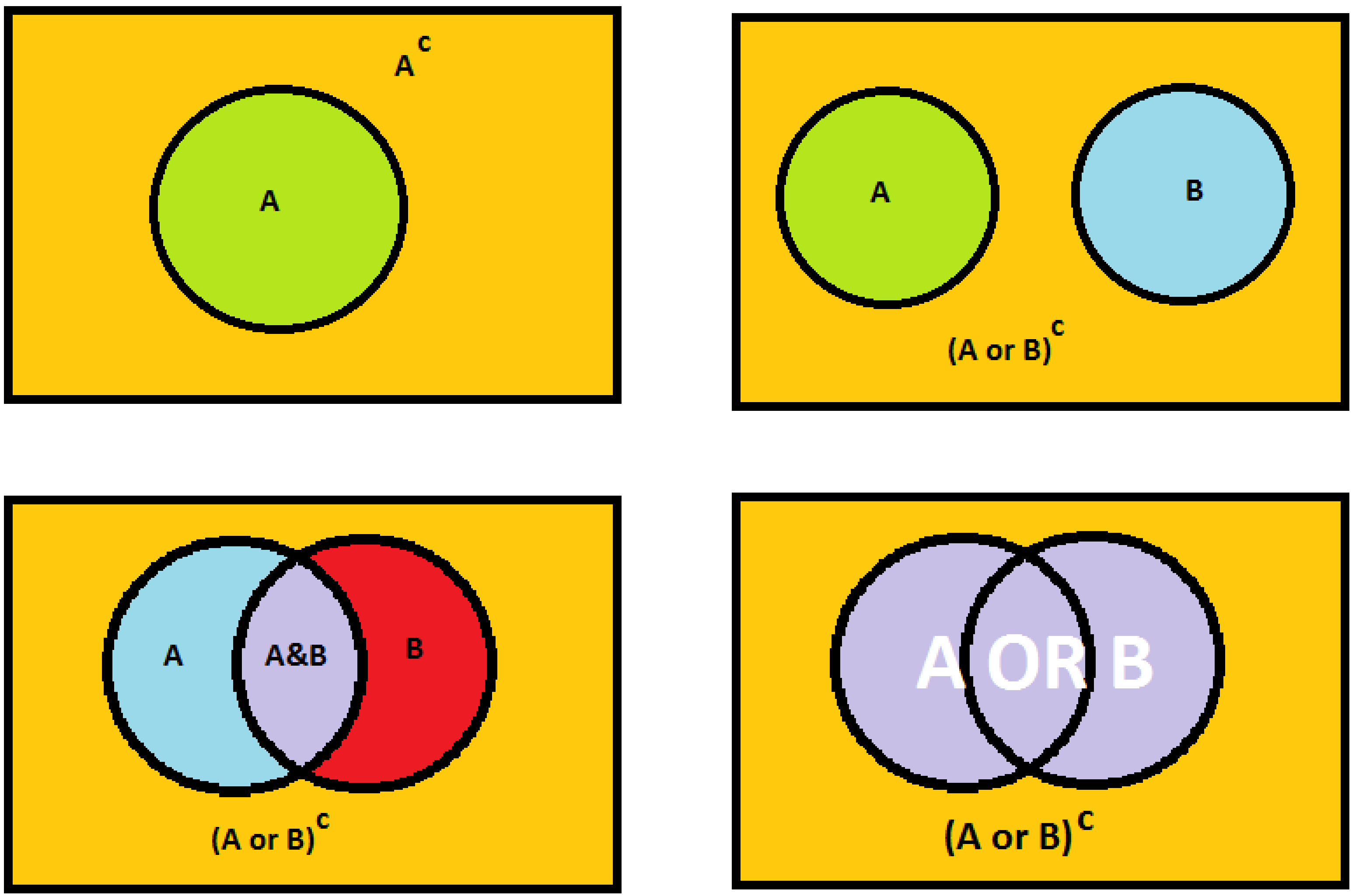
- $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$
  - The probability of A or B happening is the probability of A, plus the probability of B, minus the probability of A & B because we're double counting the probability that they both happen
  - **For mutually exclusive or disjoint events** $\boldsymbol{P(A\ or\ B) = P(A) + P(B)}$



Probability:
Additive Rule of Probability

- **Remember, in math or means one, the other or both.**
- **WITHOUT MATH:** "You can have soup or salad."
  - You could only order one
- **WITH MATH:** "You can have soup or salad."
  - You could order soup, salad or both soup and salad

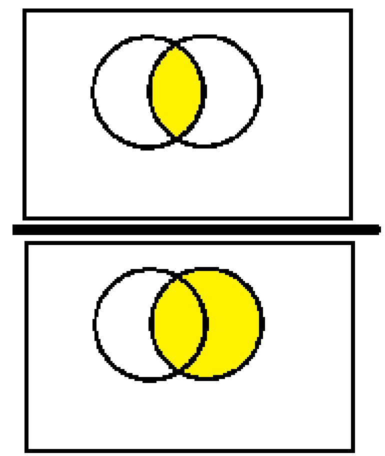
Putting it Together: Summary



Probability: Conditional Probability

- **Conditional Probabilities**
  - The probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} =$$



Probability Rules

- The probability of A and B happening is either:
  - The probability of A times the probability of B given A
  - The probability of B times the probability of A given B

- $\mathrm{P(A\ and\ B)} = \mathrm{P(A)} * \mathrm{P(B|A)}$

$$= \mathrm{P(B)} * \mathrm{P(A|}B\mathrm{)}$$

Independence

- Two events, A and B, are **independent** if the fact that A occurs does not affect the probability of B occurring.

- The idea is that knowing the outcome of one of these events doesn't give us any information about the probability of the second event.

Independence

- **Random Experiment:** Flipping a coin twice
- **Events: A=** first toss , **B=** second toss
  - **Q:** If I know that the first toss was heads, i.e. A=heads, does the probability of flipping a heads on the second flip change?
  - **A:** No, the first toss has no impact on the second toss so they are **independent**

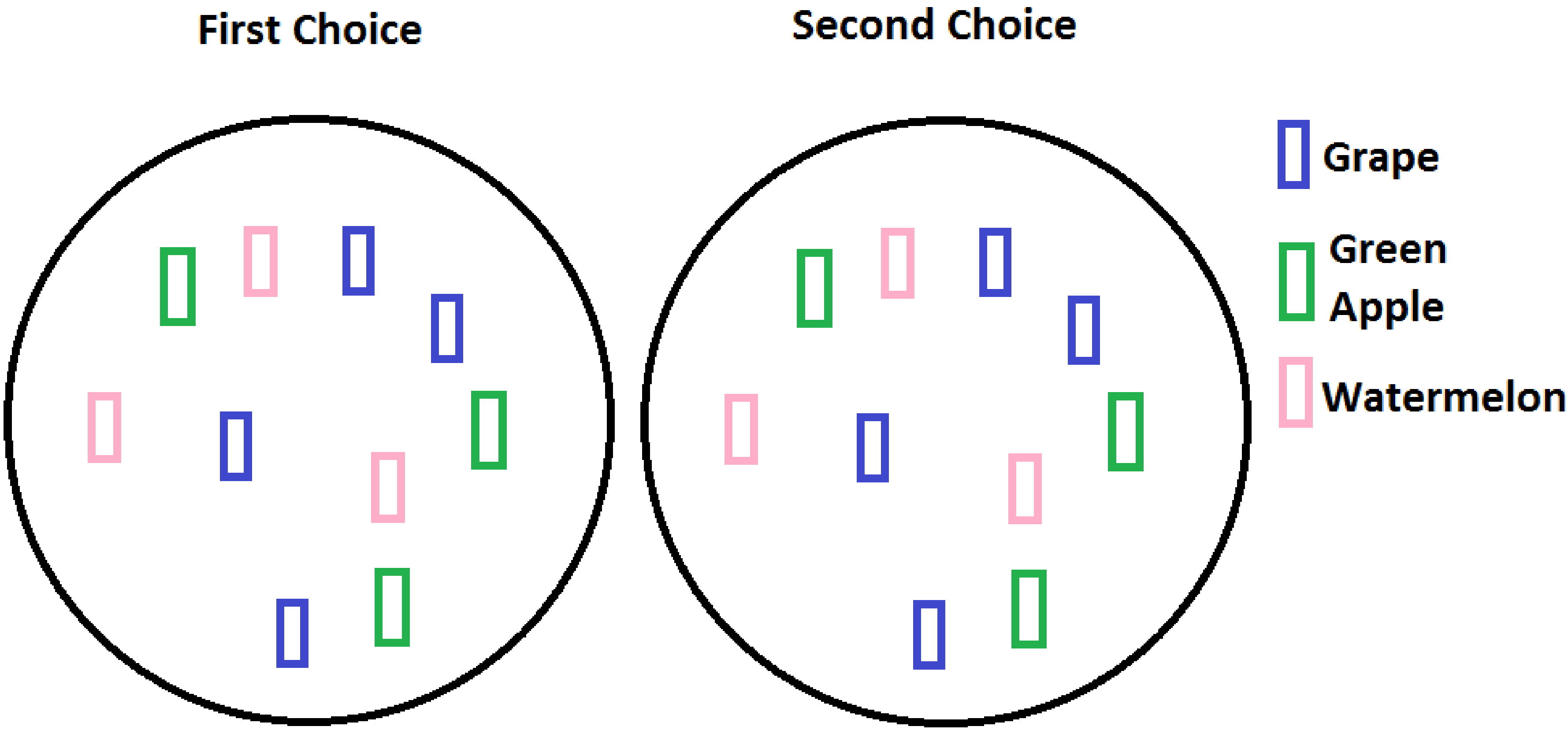
Independence

- **Random Experiment:** Picking from different flavors of Jolly Ranchers out of a bowl
- **Events: A=** first pick, **B=** second pick
  - **Q:** If I know that the first pick was watermelon, i.e. A = watermelon, does the probability of picking a watermelon on the second pick change?
  - **A1(with replacement)**: No, because we put back the first choice and we have the same exact bowl to pick our second. Now, there are the same number of watermelon Jolly Ranchers and jolly ranchers overall. This experiment would be **independent.**
    - Math if we started with ten Jolly Ranchers, three of which were watermelon: $P(A=W)=3/10=.3$ $\rightarrow P(B=W|A=W)=3/10=.3$

Independence

- **A1(with replacement)**: No, because we put back the first choice and we have the same exact bowl to pick our second. Now, there are the same number of watermelon Jolly Ranchers and jolly ranchers overall. This experiment would be **independent.**
  - If we started with ten Jolly Ranchers, three of which were watermelon:
    - $P(A=W)=3/10=.3 \rightarrow P(B=W|A=W)=3/10=.3$

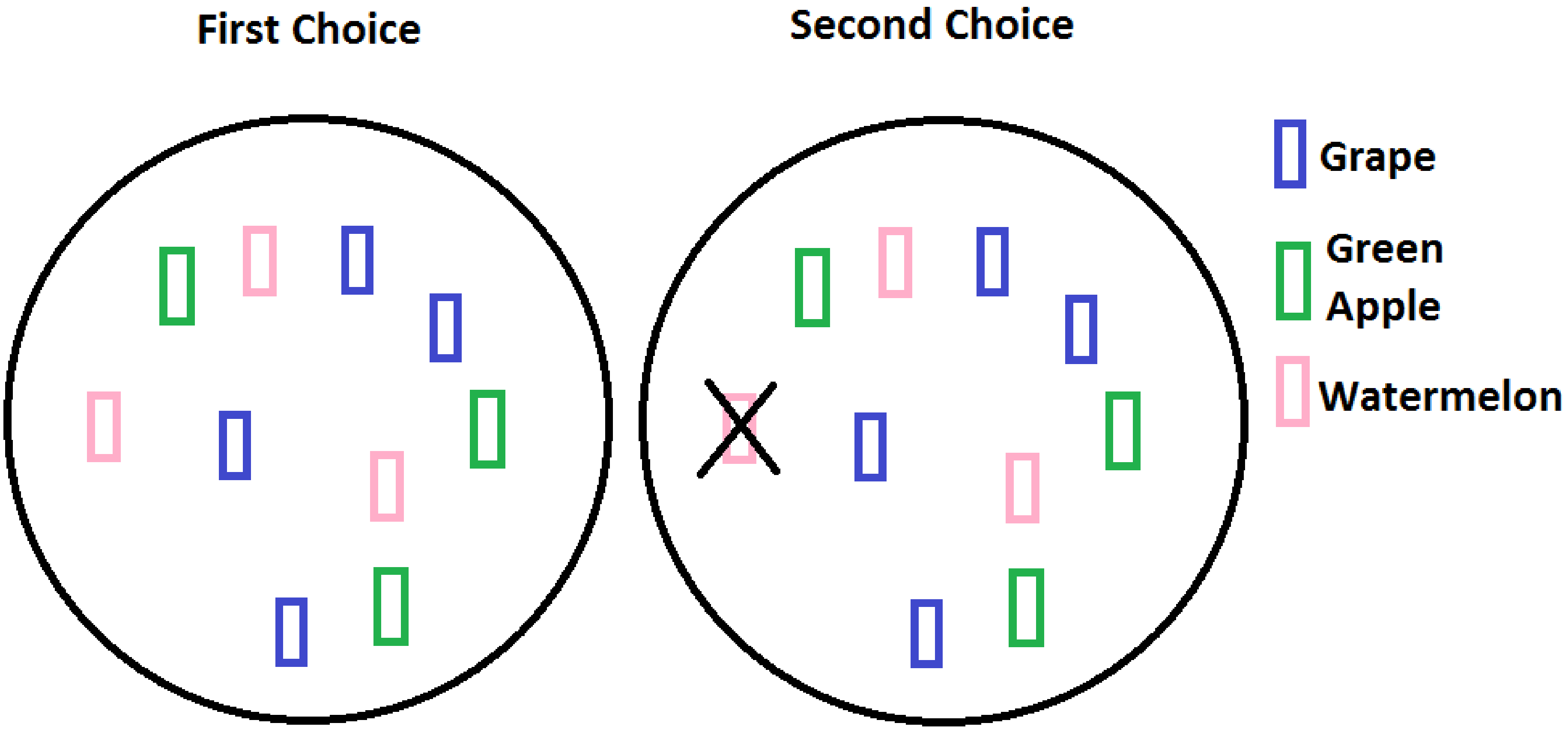
Independence



Independence

- **A1(without replacement)**: Yes, because now there are less watermelon Jolly Ranchers and less jolly ranchers overall. This experiment would not be **independent**, it would be **dependent**.
  - Math if we started with ten Jolly Ranchers, three of which were watermelon:
    - $P(A=W)=3/10=.3 \rightarrow P(B=W|A=W)=2/9=.22$

Independence



Walkthrough

- Sampling with or without Replacement
  - https://www.youtube.com/watch?v=uKTjh-6PFjo

- Conditional Probabilities
  - https://www.youtube.com/watch?v=JGeTcRfKgBo

To Check for Independence

- Two events, A and B, are independent if:
  1. If $P(A|B) = P(A)$
  2. If $P(B|A) = P(B)$
  3. If $P(A \text{ and } B) = P(A)*P(B)$

-**Note:** If any of these are true, the others are also true and the events A and B are independent

Independence

- Recall:
- $\mathrm{P(A\ and\ B)} = \mathrm{P(A)} * \mathrm{P(B|A)}$

  $$= \mathrm{P(B)} * \mathrm{P(A|}B)$$

  – **For independent events**
    - $P(A\ and\ B) = P(A) * P(B)$
    - This is because $\mathrm{P(B|A)}$=P(B) and $\mathrm{P}(A|B)$=P(A) when A and B are independent

Other Types of Events

- **Impossible:** when the probability of the event occurring is zero
- **Certain:** when the probability of the event occurring is one
- **Unusual:** when the probability of the event occurring is low. We consider low to be less than 0.05.

Partitions

- A **partition** of size k of a **Sample Space** denoted $\{A_1, A_2, \ldots A_k\}$ is a collection of k **mutually exclusive** sets such that their union makes up the Sample Space
- Note:
  - $P(A_i \cap A_j) = 0$ for all $i \neq j$
  - $A_1 \cup A_2 \cup \ldots \cup A_k = S$
    - P$(A_1 \cup A_2 \cup \ldots \cup A_k) = P(S) = 1$

Bayes's Rule

- Given a partition A on the Sample Space and event B:

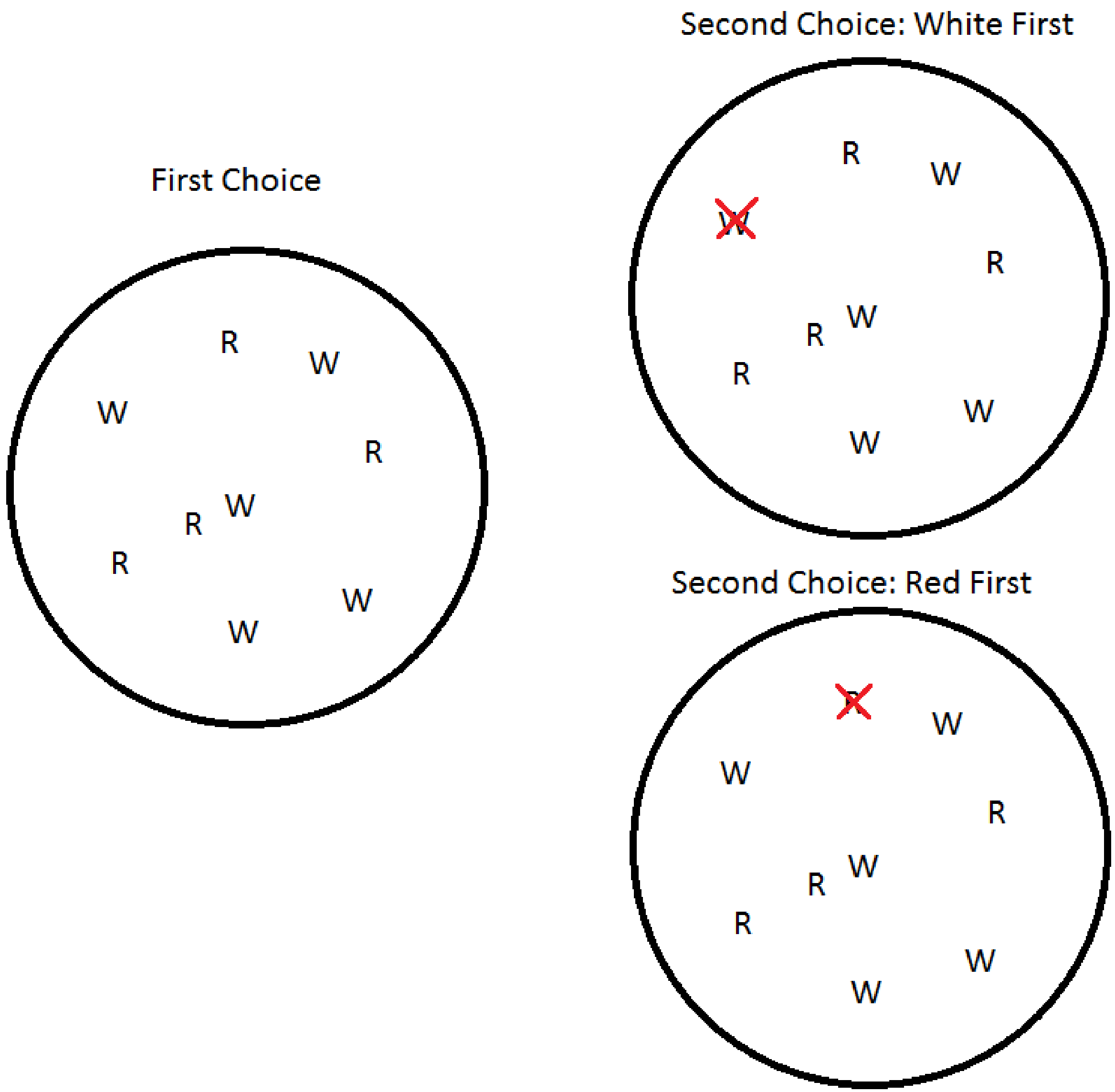
$$P(A_i|B) = \frac{P(B_i \cap A)}{P(A)}$$

$$= \frac{P(B_i) * P(A|B_i)}{[P(B_1) * P(A|B_1) + \cdots + P(B_k) * P(A|B_k)]}$$

Tree Diagram: Example

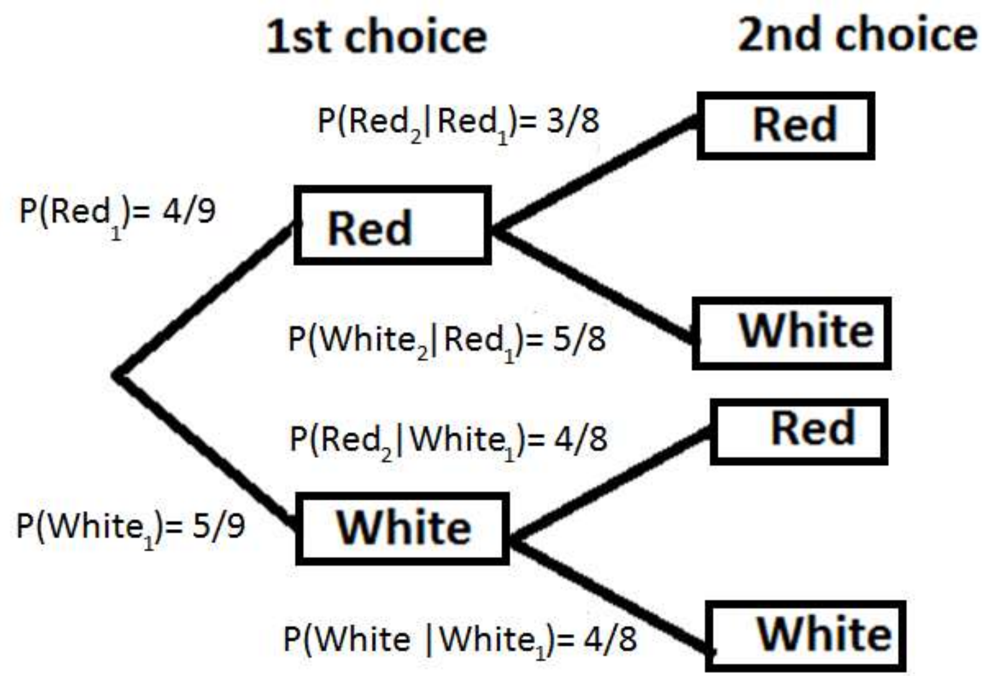
- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random:
  - One at a time **without replacement**
  - Without replacement means exactly what it sounds like – we don't put our first choice back
  - In other words, think of it this way – you choose one seed, plant it and it's gone forever, and then you choose another from the sack

Tree Diagram: Example



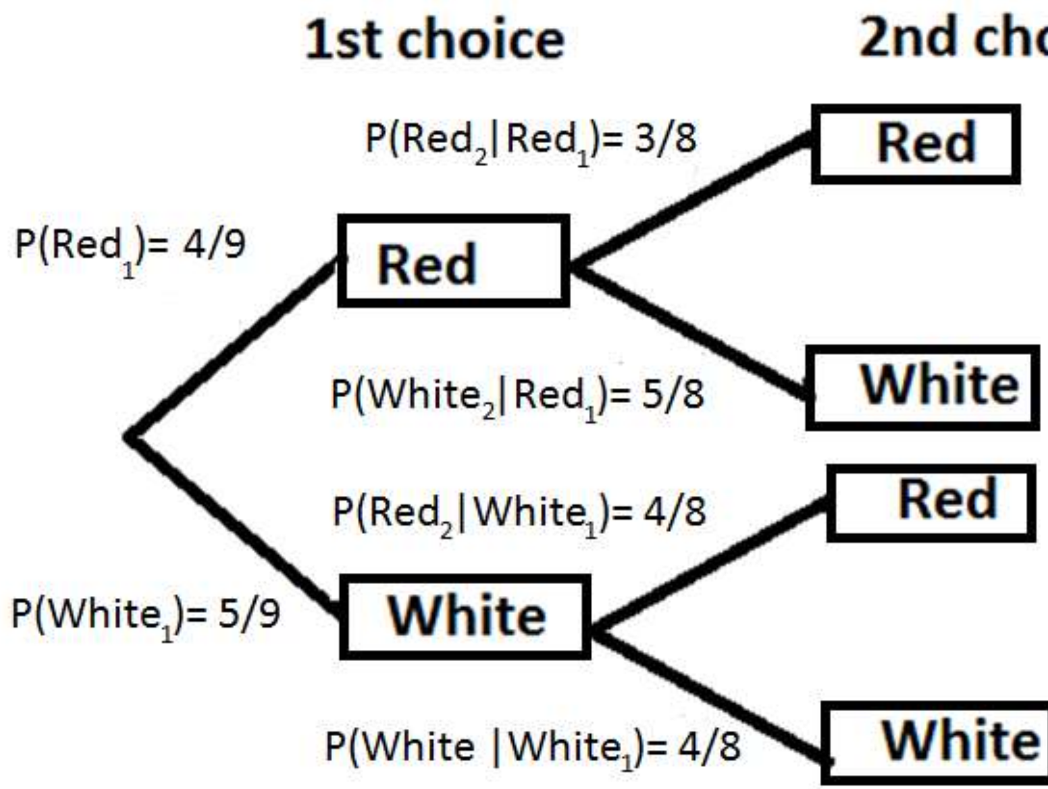
Tree Diagram: Example

- The first choice is from nine seeds
  - 4 are red
  - 5 are white



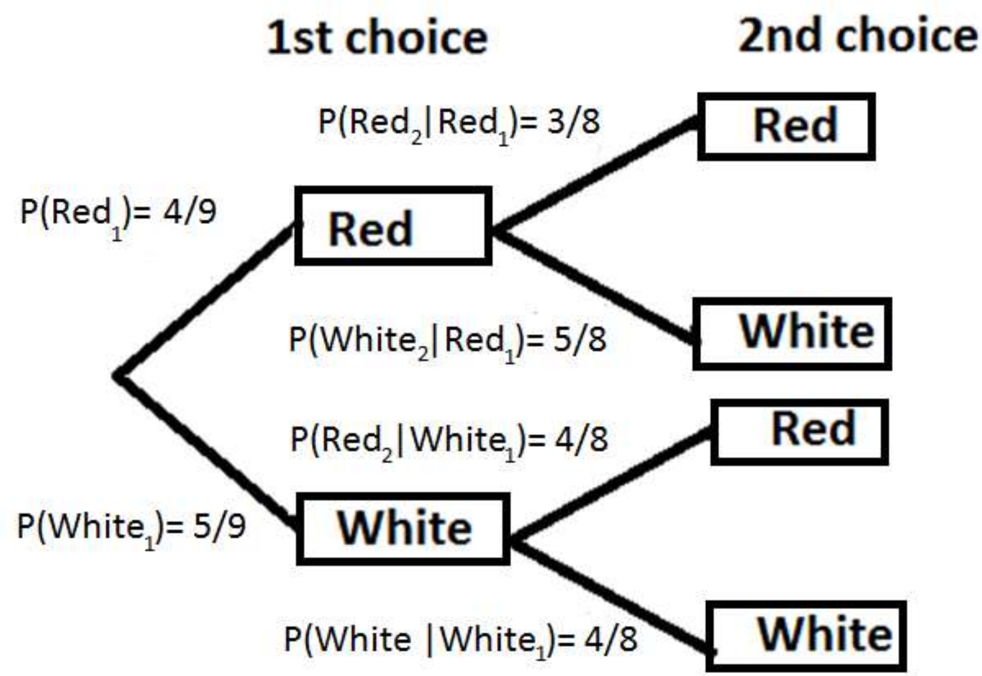
Tree Diagram: Example

- The second choice is from eight seeds because we chose one without replacement



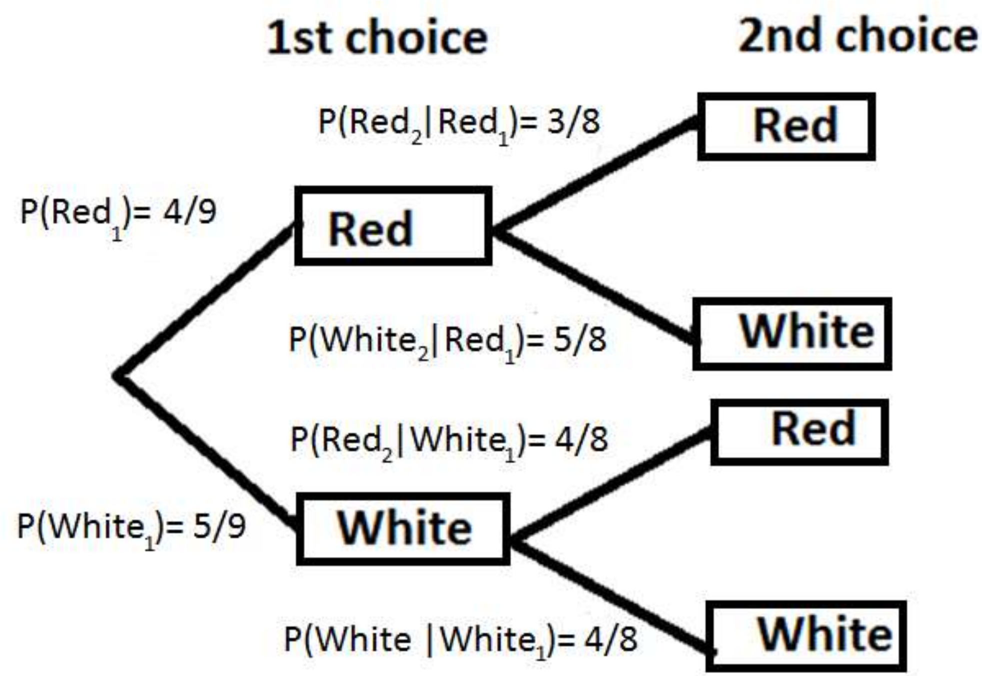
Tree Diagram: Example

- The second choice is from eight seeds
- If the first was red:
  - 3 are red
  - 5 are white



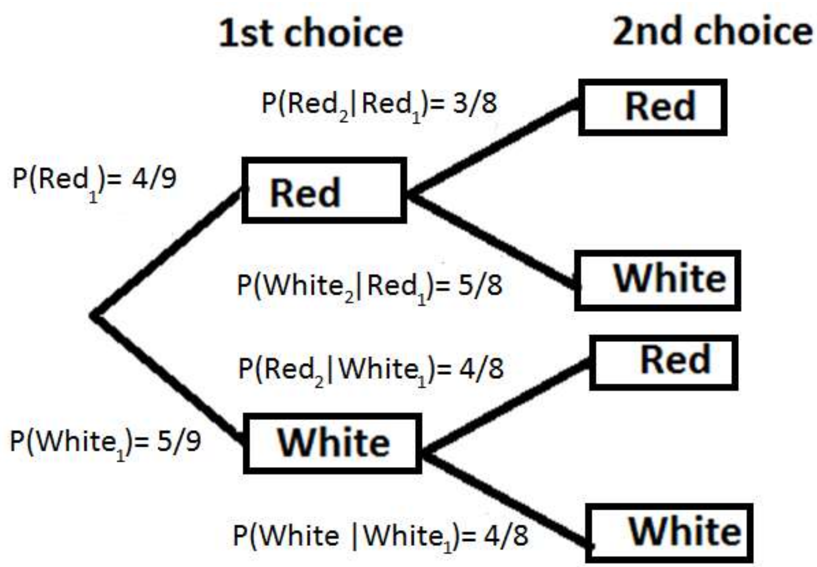
Tree Diagram: Example

- The second choice is from eight seeds
- If the first was white:
  - 4 are red
  - 4 are white



Tree Diagram: Example

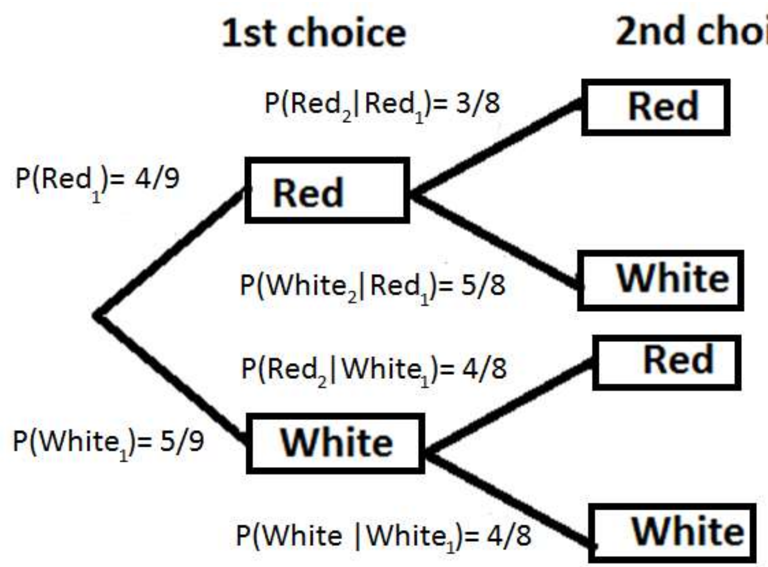
- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random without replacement



- The probability of selecting a red on our first try $P(Red_1) = \frac{4}{9}$

Tree Diagram: Example

- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random without replacement



- The probability of selecting a white on our first try
- $P(White_1) = \frac{5}{9}$

Tree Diagram: Example

- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with four red seeds and we selected a red on our first try, so now we have three red seeds

$$P(Red_2|Red_1) = \frac{3}{8}$$

Tree Diagram: Example

- The probability of selecting a red on our second try given we got a red on our first try
  – We started with nine seeds and we selected one without replacement, so now we have eight seeds
  – We started with five white seeds and we selected a red on our first try, so we still have five white seeds

$$P(White_2|Red_1) = \frac{5}{8}$$

Tree Diagram: Example

- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with four red seeds and we selected a white on our first try, so we still have four red seeds

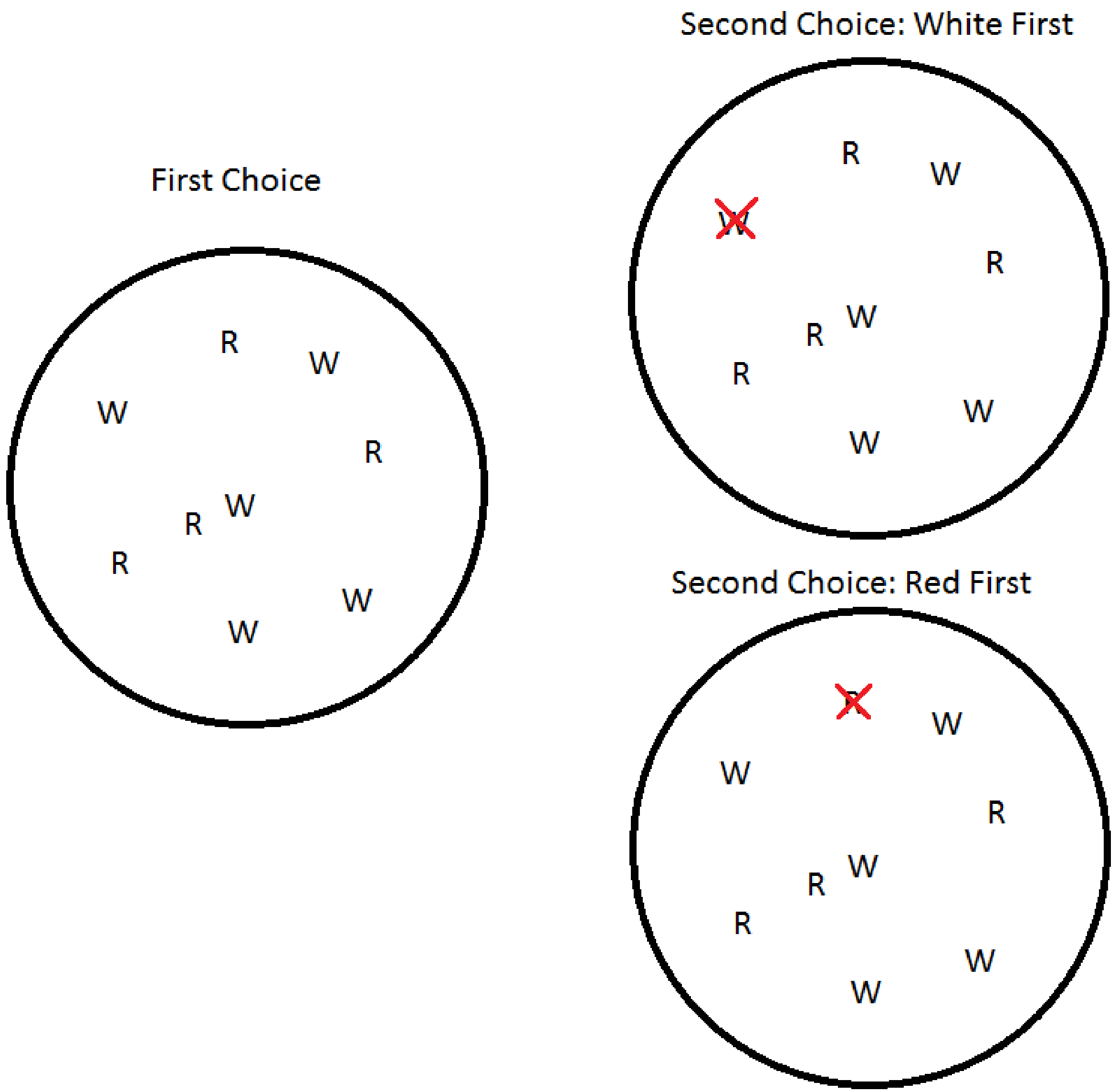
$$P(Red_2|White_1) = \frac{4}{8}$$

Example 5: Flowers

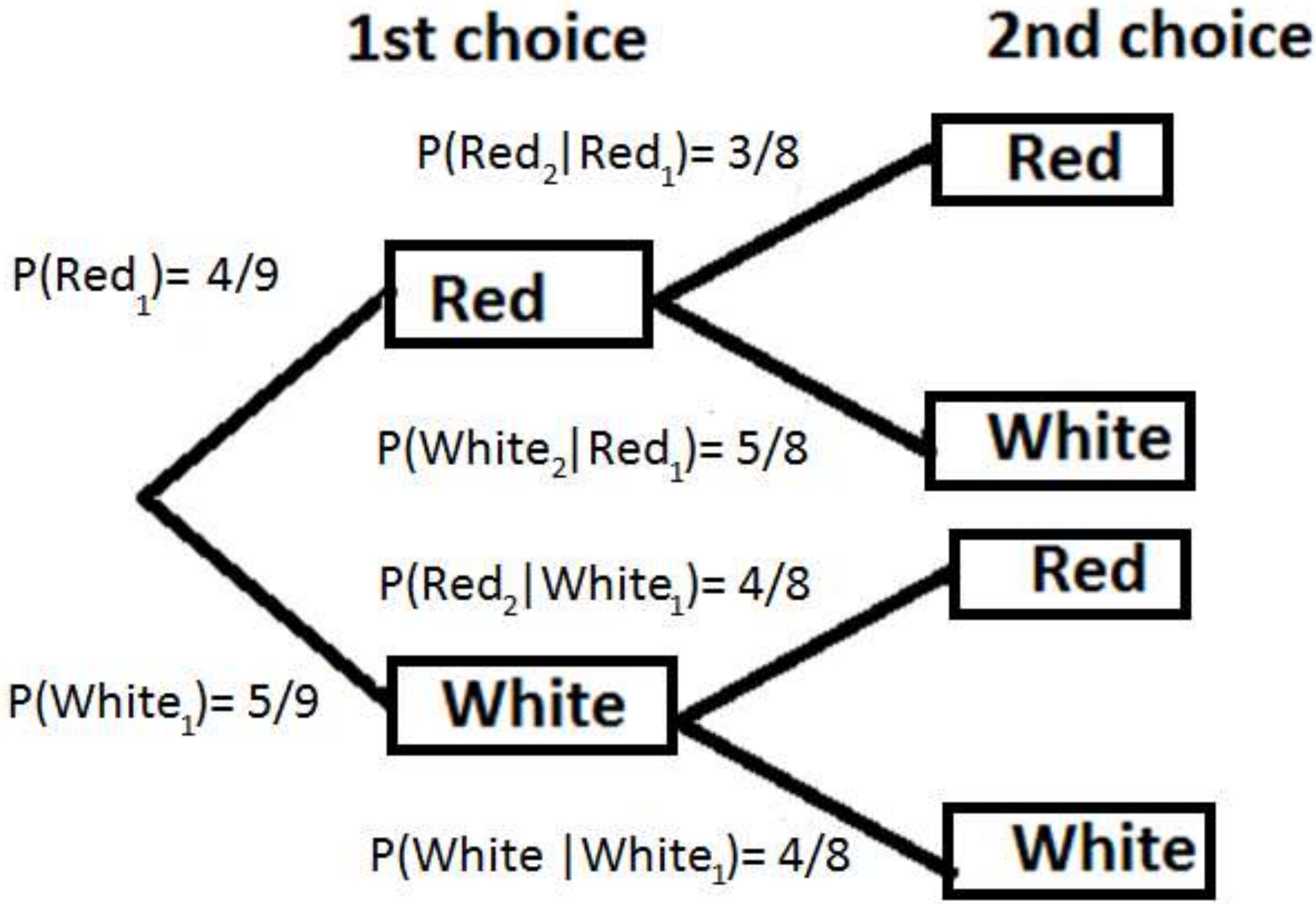
- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with five white seeds and we selected a white on our first try, so now we have four white seeds

$$P(White_2|White_1) = \frac{4}{8}$$

Tree Diagram: Example



Tree Diagram: Example



Tree Diagram: Example

- To find ‘and’ probabilities we just multiply across the branches
- Remember $\mathrm{P(A\ and\ B) = P(A) * P(B|A)}$
- The probability of choosing two red seeds in a row

$$P(R_1\ and\ R_2) = P(R_1) * P(R_2|R_1)$$
$$= \left(\frac{4}{9}\right) * \left(\frac{3}{8}\right) = \left(\frac{1}{6}\right)$$

Tree Diagram: Example

- To find 'and' probabilities we just multiply across the branches
- Remember $\mathrm{P(A\ and\ B) = P(A) * P(B|A)}$
- The probability of choosing a red seed first, and then a white seed

$$P(R_1\ and\ W_2) = P(R_1) * P(W_2|R_1)$$
$$= \left(\frac{4}{9}\right) * \left(\frac{5}{8}\right) = \left(\frac{5}{18}\right)$$

Tree Diagram: Example

- To find 'and' probabilities we just multiply across the branches
- Remember $\mathrm{P(A\ and\ B) = P(A) * P(B|A)}$
- The probability of choosing a white seed first, and then a red seed

$$P(W_1\ and\ R_2) = P(W_1) * P(R_2|W_1)$$
$$= \left(\frac{5}{9}\right) * \left(\frac{4}{8}\right) = \left(\frac{5}{18}\right)$$

Tree Diagram: Example

- To find 'and' probabilities we just multiply across the branches
- Remember $\mathrm{P(A\ and\ B)} = \mathrm{P(A)} * \mathrm{P(B|A)}$
- The probability of choosing two white seeds in a row

$$P(W1\ and\ W2) = P(W_1) * P(W_2|W_1)$$
$$= \left(\frac{5}{9}\right) * \left(\frac{4}{8}\right) = \left(\frac{5}{18}\right)$$

Tree Diagram: Example

- To find ‘and’ probabilities we just multiply across the branches
- Remember $\mathrm{P(A\ and\ B)} = \mathrm{P(A)} * \mathrm{P(B|A)}$
- The probability of choosing a white and a red, regardless of order

$$P(1\, red \,\&\, 1\, white) = P(R_1 \& W_2\ or\ W_1 \& R_2)$$
$$= P(R_1 \& W_2) + P(W_1 \& R_2)$$
$$= \left(\frac{5}{18}\right) + \left(\frac{5}{18}\right) = \left(\frac{10}{18}\right) = \left(\frac{5}{9}\right)$$

Summary!

Probability

Type	Description
Random Experiment	Any action or observable experiment where the outcome is random; possible outcomes are known but it is uncertain which will occur for any given observation.
Sample space	The set of all possible outcomes in an experiment
Event	We refer to a particular outcome occurring as an event
Law of Large Numbers	As the number of trials increase the **proportion** of occurrences of any given outcome approaches the probability in the long run.
Probability of event A $0 \leq P(A) \leq 1$	$P(A) = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$

Probability

Type	Description
Probability of event A $0 \leq P(A) \leq 1$	$P(A) = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$
Complement Rule ('not A')	$P(A^c) = 1 - P(A)$
Conditional Probability ('A given B')	$P(A\|B) = \frac{P(A\ and\ B)}{P(B)}$
Or Probability ('A or B')	$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B).$
And Probability ('A and B')	$\mathrm{P(A\ and\ B) = P(A) * P(B\|A)}$ $= \mathrm{P(B) * P(A\|}B)$

Probability

Type	Description
Independence	Two events, A and B, are **independent** if the fact that A occurs does not affect the probability of B occurring. Independent if: 1. If $P(A\|B) = P(A)$ 2. If $P(B\|A) = P(B)$ 3. If $P(A \text{ and } B) = P(A) \times P(B)$ -**Note:** If any of these are true, the others are also true and the events A and B are independent
Mutually Exclusive (disjoint)	Two events, A and B, are **mutually exclusive** if if A and B cannot happen together; if they do not have any common outcomes If $P(A\&B) = 0$
Intersection	The **intersection** of A and B consists of outcomes that are both in A and N
Union	The **union** of A and B consists of outcomes that are A or B or both

Partitions

- A **partition** of size k of a **Sample Space** denoted $\{A_1, A_2, \ldots A_k\}$ is a collection of k **mutually exclusive** sets such that their union makes up the Sample Space
- Note:
  - $P(A_i \cap A_j) = 0$ for all $i \neq j$
  - $A_1 \cup A_2 \cup \ldots \cup A_k = S$
    - P$(A_1 \cup A_2 \cup \ldots \cup A_k) = P(S) = 1$

Bayes's Rule

- Given a partition A on the Sample Space and event B:

$$P(A_i|B) = \frac{P(B_i \cap A)}{P(A)}$$

$$= \frac{P(B_i) * P(A|B_i)}{[P(B_1) * P(A|B_1) + \cdots + P(B_k) * P(A|B_k)]}$$

Other Types of Events

- **Impossible:** when the probability of the event occurring is zero

- **Certain:** when the probability of the event occurring is one

- **Unusual:** when the probability of the event occurring is low. We consider low to be less than 0.05.

Counting Summary

Type	Description	Formula
Combination	A selection of objects from a set when the order in which the objects are selected doesn't matter.	${}_nC_r = \frac{n!}{r!\,(n-r)!}$
Permutation w/replacement	A selection of objects from a set when the order in which the objects are selected matters and an object can be selected more than once.	n^r
Permutation w/o replacement	A selection of objects from a set when the order in which objects are selected matters and an object cannot be selected more than once.	${}_nP_r = \frac{n!}{(n-r)!}$
Permutation of non-distinct items	The number of ways n objects can be arranged when they are broken up by kind	$\frac{n!}{(n_1! * n_2! * \dots * n_k!)}$